② $x_1' = x_2 - x_1 - x_1^2$ R_1

$x_2' = 3x_1 - x_2 - x_1^2$ R_2

R:

$$\left.\begin{aligned} x_2 - x_1 - x_1^2 &= 0 \\ 3x_1 - x_2 - x_1^2 &= 0 \end{aligned}\right\} \quad x_1 + x_1^2 = 3x_1 - x_1^2$$

$$2x_1^2 - 2x_1 = 0$$

$$2x_1(x_1 - 1) = 0$$

$x_1 = 0$ $\qquad$ $x_1 = 1$

$x_2 = 0$ $\qquad$ $x_2 = 2$

(0,0)

Linearizovani sistem:

$$\frac{\partial f_1}{\partial x_1} = -1 - 2x_1 \xrightarrow[(0,0)]{} -1$$

$$\frac{\partial f_1}{\partial x_2} = 1$$

$x_1' = -x_1 + x_2$

$x_2' = 3x_1 - x_2$

$$A = \begin{pmatrix} -1 & 1 \\ 3 & -1 \end{pmatrix}$$

$\Downarrow$

$\lambda^2 + 2\lambda - 2 = 0$

$\lambda_{1/2} = -1 \pm \sqrt{3}$ $\Rightarrow$ imamo bar jednu nulu koja je pozitivna $\Rightarrow$

(T) $\Rightarrow$ $(x_1, x_2) = (0,0)$ je nestab. položaj

(1,2)

Prvo vršimo translaciju da bismo prebacili na (0,0)

$x_1 = y_1 + 1$

$x_2 = y_2 + 2$

$y_1' = y_2 + 2 - y_1 - 1 - (y_1 + 1)^2 = \dots = -3y_1 + y_2 - y_1^2$

$y_2' = y_1 - y_2 - y_1^2$

Lin. sistem: $\begin{cases} y_1' = -3y_1 + y_2 \\ y_2' = y_1 - y_2 \end{cases}$

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$$

$\lambda^2 + 4\lambda + 2 = 0$

$= -2 \pm \sqrt{2} < 0 \overset{T}{\Rightarrow}$ (1,2) asimp. stab. položaj ravn. za naš početni sistem

(3.) $x_1' = \ln(1 + x_2 + \sin x_1)$

$x_2' = 2 + \sqrt[3]{3\sin x_1 - 8}$

$1 + x_2 + \sin x_1 = 1 \quad (1) \qquad \ln x = 0 \iff x = 1$

$\sqrt[3]{\underbrace{3\sin x_1 - 8}_{= -8}} = -2 \quad (2)$

$\Rightarrow \sin x_1 = 0 \quad , \; x_1 = k\pi \; , \; k \in \mathbb{Z}$

$(1) \Rightarrow x_2 = 0$

$(k\pi, 0)$ - svi položaji ravnoteže

Translacija: $x_1 = y_1 + k\pi$

$x_2 = y_2$

$y_1' = \ln(1 + y_2 + \sin(y_1 + k\pi))$

$y_2' = 2 + \sqrt[3]{3\sin(y_1 + k\pi) - 8}$

$\sin(y_1 + k\pi) = \sin(y_1) \cdot \cos(k\pi) + \cos(y_1) \sin(\overset{0}{k\pi}) = (-1)^k \sin(y_1)$

$\Rightarrow y_1' = \ln(1 + \overbrace{y_2 + (-1)^k \sin y_1}^{t}) \approx y_2 + (-1)^k \sin y_1 \approx y_2 + (-1)^k y_1$

$y_2' = 2 + \sqrt[3]{3(-1)^k \sin(y_1) - 8}$

$$\sqrt[3]{3(-1)^k \sin y_1 - 8} = -2\sqrt{1 - \frac{3}{8}(-1)^k \sin y_1}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \dots$$

$$\approx -2\left(1 + \frac{1}{3}\left(-\frac{3}{8}(-1)^k \sin y_1\right)\right) = -2 + \frac{1}{4}(-1)^k \sin y_1 \approx -2 + \frac{1}{4}(-1)^k y_1$$

$$\Rightarrow y_2' = 2 + \sqrt[3]{\dots} \approx \frac{1}{4}(-1)^k y_1$$

$$A = \begin{pmatrix} (-1)^k & 1 \\ \frac{1}{4}(-1)^k & 0 \end{pmatrix}$$

$$\det(A - \lambda E) = \dots = ((-1)^k - \lambda)(-\lambda) - \frac{1}{4}(-1)^k = 0$$

$$\lambda^2 - (-1)^k \lambda - \frac{1}{4}(-1)^k = 0$$

$$\lambda_{1/2} = \frac{(-1)^k \pm \sqrt{1 + (-1)^k}}{2}$$

k-neparan $\Rightarrow \lambda_{1/2} = -\frac{1}{2}$ stabilan pol. ravn. (asimp. stab)

k-parno $\Rightarrow \frac{1 \pm \sqrt{2}}{2}$ nestabilan položaj ravn.

Za vježbu: $x_1' = x_2$ Ispitati sve položaje ravnoteže.

$x_2' = \sin(x_1 + x_2)$

Teoreme Ljapunova

$$\begin{cases} x_1' = f_1(x) \\ \vdots \\ x_n' = f_n(x) \end{cases}$$ ← dinamički sistem

Def. $V(x)$ koje su neprekidno dif. u nekoj okolini tačke $(0,0)$

i $\left.\begin{array}{l} V(x) > 0 \quad \forall x \in O(0) \\ V(0) = 0 \end{array}\right\} \Rightarrow$ poz. definitna.

Def. Poz. definitnu f-ju $V(x)$ nazivamo f-jom Ljapunova za sistem

$x' = F(x)$ ako $\forall x \in O(0)$ $\quad \dot{V}(x) = \langle \operatorname{grad} V(x), F(x) \rangle \leq 0$

Teorema: Ako u nekoj okolini $O(0)$ postoji Ljapunova f-ja za

$\{ x' = F(x)$, tada je $x = 0$ stabilan. ($x = 0$ položaj ravnoteže pretp.)

Teorema: $\{$ -11-

$\forall x \in O(0)$ $\quad \dot{V}(x) \leq -W(x)$, gdje je $W(x)$ neprekidno diferencijabilna pozitivno definitna f-ja u $O(0)$. Tada položaj ravnoteže $x = 0$ je asimptotski stabilan.

Teorema: Ako postoji $V(X)$ t.d. je $V(0)=0$, $V(X)>0$ (poz. definitna),

$\dot{V}(X) \geq W(X)$ W - poz

$\Rightarrow$ $X=0$ nestabilna.

Funkcije Ljapunova:

(4.) $x_1' = x_1^3 - x_2$

$x_2' = x_1 + x_2^3$

R//

$V(X) = \alpha x_1^2 + \beta x_2^2$

$\dot{V}(X) = 2\alpha x_1 (x_1^3 - x_2) + 2\beta x_2 (x_1 + x_2^3) =$

$= 2\alpha x_1^4 \underbrace{- 2\alpha x_1 x_2 + 2\beta x_1 x_2}_{0} + 2\beta x_2^4 =$

$\alpha = \beta = 1$

$= 2x_1^4 + 2x_2^4 = 2(x_1^4 + x_2^4) \geq 0$ nestabilno

$x_1^4 + x_2^4$

(5.) $x_1' = x_2 - x_1^3 - x_1 x_2^2$

$x_2' = -x_1 - x_2 x_1^2 - x_2^3$

$V(x_1, x_2) = \alpha x_1^2 + \beta x_2^2$

$\dot{V}(x) = 2\alpha x_1 (x_2 - x_1^3 - x_1 x_2^2) + 2\beta x_2 (-x_1 - x_2 x_1^2 - x_2^3) =$

$(\alpha, \beta) = (1, 1)$

$= -2x_1^4 - 2x_1^2 x_2^2 - 2x_2^2 x_1^2 - 2x_2^4 =$

$= -2(x_1^4 + 2x_1^2 x_2^2 + x_2^4) = -2(x_1^2 + x_2^2)^2 < 0$

$(x_1, x_2) \neq (0,0)$

$\leq -(x_1^2 + x_2^2)^2$

$x = 0$ je asimptotski stabilan

(6.) $x_1' = -x_2 - x_1^3 - x_1 x_2^2$ $\rightarrow \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$

$x_2' = 2x_1 - 2x_1^2 x_2 - 2x_2^3$

$\lambda^2 + 2 = 0$

$\lambda = \pm i\sqrt{2}$

$V(x_1, x_2) = \alpha x_1^2 + \beta x_2^2$

$\dot{V}(x) = 2\alpha x_1 (-x_2 - x_1^3 - x_1 x_2^2) + 2\beta x_2 (2x_1 - 2x_1^2 x_2 - 2x_2^3)$

$4\beta = 2\alpha$ (hoćemo da nuliramo $x_1 x_2$ jer za njih ne znamo hoće li biti poz)

$2\beta = \alpha \Rightarrow \alpha = 2$

$\beta = 1$

$V(x_1, x_2) = 2x_1^2 + x_2^2$

$= -4x_1^4 - 4x_1^2 x_2^2 - 4x_1^2 x_2^2 - 4x_2^4 = -4(x_1^4 + 2x_1^2 x_2^2 + x_2^4) =$

$= -4(x_1^2 + x_2^2)^2 < 0$

za $(x_1, x_2) \neq (0,0)$

$\leq -(x_1^2 + x_2^2)^2$

(T) $\Rightarrow$ asimptotski stabilan

položaj ravnoteže

(7.) $x_1' = -x_2 + x_1^3 + x_1 x_2^2$ $\to$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\lambda^2+1=0$

$x_2' = x_1 + x_1^2 x_2 + x_2^3$ $\lambda = \pm i$

R//

$V(x) = x_1^2 + x_2^2$

$\dot{V}(x) = 2(x_1^2 + x_2^2)^2 > 0$ za $(x_1, x_2) \neq (0,0)$

$\Rightarrow$ nestabilan položaj ravnoteže

(8.) $x_1' = -\sin x_2$

$x_2' = x_1$

R//

Γ $\dfrac{dx_2}{dx_1} = \dfrac{f_2}{f_1}$

(1) $\dfrac{dx_1}{f_1} = \dfrac{dx_2}{f_2}$

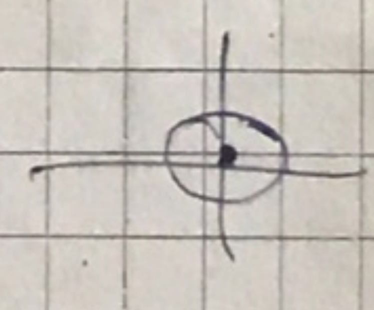
$\varphi(x_1, x_2) = c$ prvi integral

Ako postoji prvi int. za sis. (1) t.d je pozitivno definitivan, tada je položaj ravnoteže stabilan.

$\dfrac{dx_1}{-\sin x_2} = \dfrac{dx_2}{x_1}$ tzv. sistem karakteristika

$x_1\, dx_1 = -\sin x_2\, dx_2 \quad / \int$

$\dfrac{x_1^2}{2} = \cos(x_2) + c$



$\dfrac{x_1^2}{2} - \cos(x_2) = c$ prvi integral

poz difinitna

Posmatramo $V(x) = \underbrace{\dfrac{x_1^2}{2}}_{\geq 0} \underbrace{- \cos x_2 + 1}_{\geq 0}$ jer za ovo možemo da tvrdimo da je samo u $(0,0)$ nula, a u svim ostalim nije.

$\dot{V}(x) = \dfrac{2x_1}{2}(-\sin x_2) + \sin x_2 (x_1) \equiv 0$

$\dot{V}(x) \leq 0 \overset{(T)}{\Rightarrow}$ stabilan položaj ravnoteže

9. $x_1' = -x_2^3$

$x_2' = x_1^3$

R

$$\frac{dx_1}{-x_2^3} = \frac{dx_2}{x_1^3}$$

$$x_1^3\,dx_1 = -x_2^3\,dx_2 \quad /\int$$

$$\frac{x_1^4}{4} = -\frac{x_2^4}{4} + \frac{c}{4} \quad /\cdot 4$$

$x_1^4 + x_2^4 = c$ → u $(0,0)$ je nula, van te tačke $\neq 0$

$\dot{v}(x) = 4x_1^3(-x_2^3) + 4x_2^3(x_1^3) \equiv 0$, $\dot{v}(x) \leq 0 \overset{T}{\Longrightarrow}$

$(x_1, x_2) = (0,0)$ stabilan položaj ravnoteže

10. $x_1' = -2x_2 + x_2 x_3$

$x_2' = x_1 - x_1 x_3$

$x_3' = x_1 x_2$

R

$V(x) = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2$

$$\dot{v}(x) = 2\alpha_1 x_1(-2x_2 + x_2 x_3) + 2\alpha_2 x_2(x_1 - x_1 x_3) + 2\alpha_3 x_3(x_1 x_2) =$$

$$= 2(x_1 x_2(-2\alpha_1 + \alpha_2) + x_1 x_2 x_3(\alpha_1 - \alpha_2 + \alpha_3))$$

$$\left.\begin{array}{l} 2\alpha_1 = \alpha_2 \\ \alpha_1 - \alpha_2 + \alpha_3 = 0 \end{array}\right\} \quad \begin{array}{l} \alpha_1 - 2\alpha_1 + \alpha_3 = 0 \\ \alpha_1 = \alpha_3 \end{array}$$

$(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 1)$

$\dot{v}(x) \equiv 0$

$\dot{v}(x) \leq 0 \overset{T}{\Longrightarrow} (0,0,0)$ stabilan položaj ravnoteže

(11.) $x_1' = 2x_2^3 - x_1^5$

$x_2' = -x_1 - x_2^3 + x_2^5$

R:

$$V(x_1, x_2) = x_1^2 + x_2^4$$

$$\dot{V}(x_1, x_2) = 2x_1(2x_2^3 - x_1^5) + 4x_2^3(-x_1 - x_2^3 + x_2^5) =$$

$$= \cancel{4x_1x_2^3} - 2x_1^6 - \cancel{4x_2^3x_1} - 4x_2^6 + 4x_2^8 =$$

$$= -2(x_1^6 + \underbrace{2x_2^6 - 2x_2^8}_{>0}) < 0$$

$\Rightarrow$ asimp. stabilan

$x_2^6 > x_2^8$ u $O(0)$ $|x_2| < 1$

$1 > x_2^2$

$$\frac{dx_1}{2x_2^3} = \frac{dx_2}{-x_1}$$

$$x_1\,dx_1 = -2x_2^3\,dx_2 \quad / \int$$

$$\frac{x_1^2}{2} = -2\frac{x_2^4}{4} + \frac{c}{2}$$

$$x_1^2 + x_2^4 = c$$

Stabilnost partikularnog rešenja

12. $x_1' = x_2^2 - 2(t+1)x_2 - x_1$

$x_2' = 2x_1 + 2t^2 + e^{2(t-x_2)}$

$\left.\begin{array}{l} x_1 = -t^2 \\ x_2 = t \end{array}\right\}$ jedno reš.

Treba provjeriti da li je ono stabilno.

R:

$x_1 = y_1 - t^2$ <— translacija

$x_2 = y_2 + t$

$y_1' - \cancel{2t} = y_2^2 + 2y_2 t + t^2 - 2(t+1)(y_2+t) - y_1 + t^2 =$

$= y_2^2 + \cancel{2y_2 t} + \cancel{t^2} - \cancel{2t y_2} - \cancel{2t^2} - 2y_2 - \cancel{2t} - y_1 + \cancel{t^2}$

$\boxed{y_1' = -y_1 - 2y_2 + y_2^2} \approx -y_1 - 2y_2$

$y_2' + 1 = 2y_1 - 2t^2 + 2t^2 + e^{2(-y_2)}$

$\boxed{y_2' = 2y_1 + e^{-2y_2} - 1} \approx 2y_1 + \cancel{1} - 2y_2 - \cancel{1}$

$$A = \begin{pmatrix} -1 & -2 \\ 2 & -2 \end{pmatrix}$$

$\det(A - \lambda E) = (-1-\lambda)(-2-\lambda) + 4 = \lambda^2 + 3\lambda + 2 + 4 = \lambda^2 + 3\lambda + 6$

C

$\frac{-3 \pm \sqrt{9-36}}{2}$

Re

T => asimp. stabilan